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ON THE UNIFORM COMPLETE CONVERGENCE OF DENSITY FUNCTION ESTIMATES

by

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ABSTRACT

Let f be a continuous density function which has compact support [a, b]. Let W be a nonnegative weight function which is continuous on its compact support [a, b] and $\int_a^b W(x)dx = 1$. The complete convergence of

$$\sup_{-\infty<s<\infty}\left|\frac{1}{nb(n)}\sum_{k=1}^{n} W\left(\frac{s-X_k}{b(n)}\right) - f(s)\right|$$

to zero is obtained under varying conditions on the bandwidths b(n) and smoothness of W. For example, one choice of the weight function is Epanechnikov's optimal function and $nb^2(n) > n^\delta$ for some $\delta > 0$. The uniform complete convergence of the mode estimate is also considered.

Key Words and Phrases. Density function estimates, weight function, bandwidth, complete convergence, and subGuassian random variables.

1. Introduction and Preliminaries. The construction of a family of estimates of a density function $f(x)$ and of the mode has been studied by several people. Rosenblatt (1956) considered a general class of density estimates:

$$f_n(x) = \frac{1}{nb(n)} \sum_{i=1}^{n} W\left(\frac{x - X_i}{b(n)}\right), \tag{1.1}$$

where $X_1, \ldots, X_n$ are i.i.d. random variables with continuous density function $f(x)$, $W(x)$ is a bounded integrable weight function such that

$$\int_{-\infty}^{\infty} W(x)dx = 1$$

and $b(n)$ is a bandwidth that tends to zero as $n \to \infty$. Thus, the question arises as to suitable choices of $W(x)$ and $b(n)$ so that the estimate function $f_n(x)$ is optimal (in some sense). The local properties of the estimate function in (1.1) have been studied extensively [see Rosenblatt (1971) for a general survey], and a global measure of deviation of the curve $f_n(x)$ from $f(x)$ by

$$||f_n - f||_\infty = \sup_{x \in R} |f_n(x) - f(x)| \tag{1.2}$$

has been considered. Parzen (1962) showed that if the (true) underlying density function $f(x)$ is uniformly continuous then $||f_n - f||_\infty$ converges in probability to zero under the following conditions:

(P1) $\phi_W(t) = \int_{-\infty}^{\infty} e^{itx} W(x)dx$ is absolutely integrable,

(P2) $\sqrt{n}b(n) \to \infty$ as $n \to \infty$.

The results of Nadaraya (1965) and Woodroofe (1967) on uniform consistency in the strong sense are listed in the last section for comparison with the results of this paper.

The major results of this paper give a new class of "good" weight functions [which includes the optimal function] under mild conditions on the bandwidth sequence $b(n)$ where uniform consistency of the estimate $f_n(x)$ is obtained by the complete convergence [see Stout (1966)] of $||f_n - f||_\infty$ to zero. The main tools used in obtaining these results will be the smoothness of the weight function and sub-Gaussian techniques.

Throughout this paper, attention will be restricted to a density function which has compact support $[a, b]$ and weight functions $W(x)$ which satisfy

(i) $\int_a^b W(x)dx = 1$ and

(ii) $W(x)$ is nonnegative and continuous on $[a, b]$ and vanish outside $[a, b]$.

Let U_n be a polygonal approximating function on the space of continuous functions with domain $[a, b], C[a, b]$. That is,

$$g\left(a + \frac{(b - a)i}{n}\right) = [U_n(g)]\left(a + \frac{(b - a)i}{n}\right)$$

for $i = 0, 1, \ldots, n$ and $g \in C[a, b]$, and U_n is linear between the points $a + \frac{(b - a)i}{n}$ and $a + \frac{(b - a)(i + 1)}{n}$. Recall that the modulus of continuity, $\omega_g(\delta)$, is defined by Billingsley (1968) by

$$\omega_g(\gamma) = \sup_{|t-s|\le\gamma} |g(t) - g(s)| \tag{1.3}$$

for $\gamma > 0$, $s, t \in [a, b]$, and $g \in C[a, b]$.

<u>Definition [Chow (1966)]</u> A random variable X is said to be <u>sub-Gaussian</u> if there exists $\alpha \ge 0$ such that

$$E[\exp(tX)] \le \exp\left(\frac{\alpha^2 t^2}{2}\right) \text{ for all } t \in R. \tag{1.4}$$

If X is sub-Gaussian, then let

$$\tau(X) = \inf\{\alpha \geq 0\colon \text{ Inequality (1.4) holds}\}.$$

Some basic properties on sub-Gaussian random variables include:

(1) If $P[|X| \leq K] = 1$ and $EX = 0$, then $E[\exp(tX)] \leq \exp(K^2 t^2)$. (1.5)

(2) If $\tau(X) = \alpha$, then $P[|X| \geq \lambda] \leq 2 \exp(-\lambda^2/2\alpha^2)$. (1.6)

(3) The sum of two independent sub-Gaussian random variables is sub-Gaussian.

Finally, a sequence of random variables $\{X_n\}$ is said to converge completely to a random variable X if

$$\sum_{n=1}^{\infty} P[|X_n - X| > \varepsilon] < \infty \tag{1.7}$$

for each $\varepsilon > 0$. Thus, complete convergence implies convergence with probability one by Borel's inequality.

2. Main Results. In this section the complete convergence of $||f_n - f||_\infty$ to zero is obtained under conditions on the modulus of continuity of the weight function W(x) and the rate of convergence to zero by the bandwidth b(n). Also, the uniform consistency of the mode estimate is obtained in this setting. The uniform consistency of the estimate $f_n(x)$ [in the complete sense] is accomplished by two lemmas.

Lemma 1. If (i) $nb^2(n) > n^\delta$ for some $\delta > 0$ and (ii) $\omega_W\left(\frac{2b - 2a}{n^r b(n)}\right) = o(b(n))$ for some integer $r > 0$, then

$$\sup_{-\infty<s<\infty}\left|f_n(s) - \frac{1}{b(n)} EW\left(\frac{s - X_1}{b(n)}\right)\right| \to 0 \tag{2.1}$$

completely as $n \to \infty$ where $f_n(x)$ is defined in (1.1).

Proof: Since W and f_{X_1} vanish outside [a, b] and $b(n) \to 0$, the sup in (2.1) need only be taken over [2a, 2b] ([a, 2b] suffices if $a > 0$). Let $\delta_n = \frac{2(b - a)}{n^r b(n)}$ and let $\tilde{W}_k(s) = W\left(s - \frac{X_k}{b(n)}\right) - EW\left(s - \frac{X_1}{b(n)}\right)$ for each $k = 1, \ldots, n$. Thus, $E\tilde{W}_k(s) = 0$ for each $s \in [a, b]$ and each k. Furthermore,

$$\begin{aligned}\omega_{\tilde{W}_k}(\delta_n) &= \sup_{|t-s|\le\delta_n} |\tilde{W}_k(s) - \tilde{W}_k(t)| \\ &\le \sup_{|t-s|\le\delta_n}\left|W\left(s - \frac{X_k}{b(n)}\right) - W\left(t - \frac{X_k}{b(n)}\right)\right| \\ &+ \sup_{|t-s|\le\delta_n}\left|EW\left(s - \frac{X_k}{b(n)}\right) - EW\left(t - \frac{X_k}{b(n)}\right)\right| \le 2\omega_W(\delta_n).\end{aligned} \tag{2.2}$$

Hence, $\omega_{\tilde{W}_k}(\delta_n) \le 2\omega_W(\delta_n) = o(b(n))$ for each k from condition (ii). For $\epsilon > 0$ let

$$A_n = [\sup_{2a\le s\le 2b}\left|\frac{1}{nb(n)}\sum_{k=1}^n \tilde{W}_k\left(\frac{s}{b(n)}\right)\right| > \varepsilon]$$

$$= [\max_{1\le i\le n^r}\sup_{s\varepsilon I_i}\left|\frac{1}{nb(n)}\sum_{k=1}^n \tilde{W}_k\left(\frac{s}{b(n)}\right)\right| > \varepsilon] \tag{2.3}$$

where $I_i = [t_{i-1}, t_i]$ with $t_i = 2a + \frac{2i(b-a)}{n^r}$ for $i = 1, \ldots, n^r$. Hence,

$$A_n \subset [\max_{1\le i\le n^r}\left|\frac{1}{nb(n)}\sum_{k=1}^n \tilde{W}_k\left(\frac{t_i}{b(n)}\right)\right|$$

$$+ \max_{1\le i\le n^r}\sup_{s\varepsilon I_i}\left|\frac{1}{nb(n)}\sum_{k=1}^n [\tilde{W}_k\left(\frac{s}{b(n)}\right) - \tilde{W}_k\left(\frac{t_i}{b(n)}\right)]\right| > \varepsilon]. \tag{2.4}$$

However,

$$\max_{1\le i\le n^r}\sup_{s\varepsilon I_i}\left|\frac{1}{nb(n)}\sum_{k=1}^n [\tilde{W}_k\left(\frac{s}{b(n)}\right) - \tilde{W}_k\left(\frac{t_i}{b(n)}\right)]\right| \le \frac{2}{b(n)}\,\omega_W(\delta_n). \tag{2.5}$$

Since $\omega_W(\delta_n) = o(b(n))$ by condition (ii), there exists $N(r)$ such that

$$A_n \subset [\max_{1\le i\le n^r}\left|\frac{1}{nb(n)}\sum_{k=1}^n \tilde{W}_k\left(\frac{t_i}{b(n)}\right)\right| > \frac{\varepsilon}{2}]$$

for all $n \ge N(r)$. Using the basic properties of sub-Gaussian random variables $[\{\tilde{W}_k(t_i): k = 1, 2, \ldots\}$ for each $i]$, for each $n \ge N(r)$

$$P(A_n) \le P[\max_{1\le i\le n^r}\left|\frac{1}{nb(n)}\sum_{k=1}^n \tilde{W}_k\left(\frac{t_i}{b(n)}\right)\right| > \frac{\varepsilon}{2}]$$

$$\le \sum_{i=1}^{n^r} P[\left|\frac{1}{nb(n)}\sum_{k=1}^n \tilde{W}_k\left(\frac{t_i}{b(n)}\right)\right| > \frac{\varepsilon}{2}]$$

$$\le n^r 2\exp[-\varepsilon^2/4||W||_\infty B_n] \tag{2.6}$$

where $||W||_\infty = \sup_s |W(s)|$ and $B_n = \sum_{k=1}^n \left(\frac{1}{nb(n)}\right)^2 = \frac{1}{nb^2(n)}$. To obtain the complete convergence in (2.1), consider

$$\sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{N(r)} P(A_n) + \sum_{n=N(r)+1}^{\infty} P(A_n)$$

$$\le N(r) + \sum_{n=N(r)+1}^{\infty} 2n^r \exp\left(\frac{-\varepsilon^2 nb^2(n)}{4||W||_\infty}\right)$$

$$\le N(r) + \sum_{n=N(r)+1}^{\infty} 2n^r \exp(-cn^\delta) \tag{2.7}$$

where $c = \varepsilon^2/4||W||_\infty$. Thus, the series in (2.7) converges by the integral test.

Lemma 2. If the underlying density, f, is continuous, then

$$\sup_{-\infty<s<\infty} \left|\frac{1}{b(n)} EW\left(\frac{s - X_1}{b(n)}\right) - f(s)\right| \to 0 \quad \text{as } n \to \infty. \tag{2.8}$$

Proof: Since f is continuous and $\lim_{x\to\infty} f(x) = 0 = \lim_{x\to-\infty} f(x)$, given $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(x')| < \varepsilon$ whenever $|x - x'| < \delta$. Let N be sufficiently large so that $|b(n)y| < \delta$ for all $n \ge N$ and $y \in [a, b]$. Since $W(y) = 0$ for $y \notin [a, b]$,

$$\left|\frac{1}{b(n)} EW\left(\frac{s - X_1}{b(n)}\right) - f(s)\right| = \left|\frac{1}{b(n)} \int_{-\infty}^{\infty} W\left(\frac{s - x}{b(n)}\right) f(x)dx - f(s)\right|$$

$$= \left|\int_{-\infty}^{\infty} W(y)f(s - b(n)y)dy - f(s)\right| = \left|\int_a^b W(y)[f(s - b(n)y) - f(s)]dy\right|$$

$$< \varepsilon \int_a^b W(y)dy = \varepsilon \tag{2.9}$$

uniformly in s for all $n \ge N$. Hence,

$$\sup_{-\infty<s<\infty} \left|\frac{1}{b(n)} EW\left(\frac{s - X_1}{b(n)}\right) - f(s)\right| \to 0$$

as $n \to \infty$.

Thus, the proof of Theorem 1 is immediate from Lemmas 1 and 2 since for each $\varepsilon > 0$

$$P[\sup_{-\infty<s<\infty}\left|\frac{1}{nb(n)}\sum_{k=1}^{n}W\left(\frac{s-X_k}{b(n)}\right) - f(s)\right| > \varepsilon]$$

$$\le P[\sup_{-\infty<s<\infty}\left|\frac{1}{nb(n)}\sum_{k=1}^{n}\left(W\left(\frac{s-X_k}{b(n)}\right) - EW\left(\frac{s-X_1}{b(n)}\right)\right)\right| > \frac{\varepsilon}{2}]$$

$$+ P[\sup_{-\infty<s<\infty}\left|\frac{1}{b(n)}EW\left(\frac{s-X_1}{b(n)}\right) - f(s)\right| > \frac{\varepsilon}{2}] \tag{2.10}$$

and each of the terms in (2.10) is a convergent series in n. All of the conditions will be stated in Theorem 1 for easy reference.

Theorem 1: Let $\{X_n\}$ be independent random variables with the same density function $f(s)$ which is continuous and has compact support. Let $W(x)$ be a nonnegative weight function which is continuous on its compact support and integrates to 1. If

(a) $nb^2(n) > n^\delta$ for some $\delta > 0$, and

(b) $\omega_W\left(\frac{2b-2a}{n^r b(n)}\right) = o(b(n))$ for some integer $r > 0$, then

$$\sup_{-\infty<s<\infty}\left|\frac{1}{nb(n)}\sum_{k=1}^{n}W\left(\frac{s-X_k}{b(n)}\right) - f(s)\right| \to 0$$

completely as $n \to \infty$.

The case of the density function being discontinuous off its support is not entirely excluded in Theorem 1. The following steps indicate modifications which allows the theory to include a large class of density functions [for example, the uniform densities].

Step 1. For an unknown density function which is continuous on [a, b] and vanishes outside [a, b], there is no change in Lemma 1.

Step 2. In Lemma 2 it is easy to verify that for each $\varepsilon > 0$

$$\sup_{a+\varepsilon\le s\le b-\varepsilon}\left|E\frac{1}{b(n)}W\left(\frac{s-X_1}{b(n)}\right)-f(s)\right|\to 0$$

as $n\to\infty$.

Step 3. Combining steps 1 and 2, for each $\varepsilon > 0$

$$\sup_{a+\varepsilon\le s\le b-\varepsilon}\left|\frac{1}{nb(n)}\sum_{k=1}^{n}W\left(\frac{s-X_k}{b(n)}\right)-f(s)\right|\to 0$$

completely as $n\to\infty$.

Hence, the complete convergence of the maximal deviation of the density estimate holds on arbitrary closed intervals inside of [a, b]. Similar consideration was also given in Woodroofe (1967).

In Lemma 1 the modulus of continuity was used only to replace $f_n(s)$ by a polygonal approximation. Thus, the following corollary can be obtained with basically the same proof.

Corollary 1: Let the density function $f(s)$ be as stated in Theorem 1. Let $W(x)$ be a nonnegative weight function which has compact support and integrates to 1. If

(a) $nb^2(n) > n^\delta$ for some $\delta > 0$, and

(b') $\sup_{a\le s\le b}|W(s)-U_{nr}(W)(s)| = o(b(n))$, then

$$\sup_{-\infty<s<\infty}\left|\frac{1}{nb(n)}\sum_{k=1}^{n}W\left(\frac{s-X_k}{b(n)}\right)-f(s)\right|\to 0$$

completely as $n\to\infty$.

The condition $nb^2(n) > n^\delta$ need not hold for all n but only eventually. Also, the condition can be stated as

$$\text{(a')} \quad \int_d^\infty x^r \exp(-cxb^2(x))dx < \infty$$

for some $d > 0$ where $b(x)$ is a function which generates the bandwidths $b(1), b(2), \ldots$ and c is a constant.

In considering mode estimates, assume that the continuous density function $f(s)$ has a unique mode θ, that is,

$$f(\theta) = \max_{-\infty<s<\infty} f(s).$$

The sample mode θ_n is also assumed to uniquely satisfy

$$f_n(\theta_n) = \max_{-\infty<s<\infty} f_n(s) \text{ for each } n.$$

Theorem 2. If the regularity conditions of Theorem 1 or Corollary 1 (or condition (b')) are satisfied, then

$$|\theta_n - \theta| \to 0$$

completely as $n \to \infty$.

Proof: Since $f(s)$ is uniformly continuous and has a unique mode θ, for $\varepsilon > 0$ there exists $\eta > 0$ such that $|x - \theta| \geq \varepsilon$ implies that $|f(\theta) - f(x)| \geq \eta$. Thus, it suffices to show that $f(\theta_n) \to f(\theta)$ almost surely. But,

$$\begin{aligned} |f(\theta_n) - f(\theta)| &\leq |f(\theta_n) - f_n(\theta_n)| + |f_n(\theta_n) - f(\theta)| \\ &\leq \sup_{-\infty<s<\infty} |f(s) - f_n(s)| + |\max_{-\infty<s<\infty} f_n(s) - \max_{-\infty<s<\infty} f(s)| \\ &\leq 2 \sup_{-\infty<s<\infty} |f_n(s) - f(s)| \end{aligned} \tag{2.11}$$

pointwise for each n. From (2.11) and the complete convergence of $||f_n - f||_\infty$, it follows that $|\theta_n - \theta| \to 0$ completely.

3. Comparisons and Useful Weight Functions. Brief comments on Nadaraya's (1965) and Woodroofe's (1967) conditions and on useful weight functions which satisfy the results of the paper are listed for comparison.

To obtain a strong law rather than uniform consistency in probability (as listed in the Introduction), the conditions on the weight function and bandwidth sequence are expected to be more stringent. For example,

(A) let f be an unknown continuous density function. If

(N1) $W(x) = W(-x)$ is of bounded variation and $\int_{-\infty}^{\infty} x^2 W(x)dx$ exists; and

(N2) $\sum_{n=1}^{\infty} \exp(-rnb^2(n))$ exists for each $r > 0$, then

$||f_n - f||_\infty \to 0$ with probability one [Nadaraya (1965)].

Next,

(B) let the density have support on some neighborhood of compact interval, e.g., [-1, 1]. If

(W1) $W \in LIP(\beta)$, $0 < \beta \le 1$; and

(W2) $b(n)^{-r} = o(n)$ and $n = o(b(n)^{-\delta})$ with $1 < r < \delta$, then

$$\sup_{-1\le x\le 1} \left| \frac{1}{nb(n)} \sum_{k=1}^{n} W\left(x, \left(\frac{x - X_k}{b(n)}\right)\right) - f(x) \right| = o(b(n)^{\frac{1}{2}}) \text{ with probability one}$$

where

$$\sup_{-\infty<x<\infty} \int_{|y|\ge t} |y| W(x, y)dy \to 0 \text{ as } t \to \infty$$

[Woodroofe (1967)].

For the results of this paper, the weight function W(x) needed to be continuous on its compact support and satisfy a smoothness condition [(b) or (b')]. Some useful weight functions for these results will now be listed. First, Epanechnikov's (1969) optimal weight function

$$W(x) = \begin{cases} \frac{3}{4\sqrt{5}} (1 - \frac{x^2}{5}) & \text{if } |x| \le 5 \\ 0 & \text{otherwise} \end{cases}$$

can be used. In this case, let $a = -5$ and $b = 5$. Then, $|W(x) - W(y)| \le C|x - y|$ and $\omega_W\left(\frac{2b - 2a}{b(n)n^r}\right) \le C'\frac{1}{b(n)n^r}$ for constants C and C'. Thus, condition (b) is easily satisfied.

If the weight function W satisfies a Lipschitz condition of order α, then $|W(x) - W(y)| \le M|x - y|^\alpha$ and

$$\omega_W\left(\frac{1}{b(n)n^r}\right) \le M\left(\frac{1}{b(n)n^r}\right)^\alpha$$

for some $M > 0$. Thus, the bandwidth sequence $b(n)$ must be chosen so that

$$b^{\alpha+1}(n)n^{\alpha r} \to \infty \text{ as } n \to \infty \text{ for some integer } r > 0. \tag{3.1}$$

<u>Case 1: $-1 \le \alpha \le 0$.</u> No bandwidth exists for (3.1).

<u>Case 2: $\alpha > 0$.</u> If $b(n) = n^{-p}$ for some $p > 0$, then r is an integer $\ge 2p(1 + \alpha)/\alpha$. Then, $b^{1+\alpha}(n)n^{\alpha r} \ge n^{p(1+\alpha)} \to \infty$, and (3.1) is satisfied.

<u>Case 3: $\alpha < -1$.</u> Again, if $b(n) = n^{-p}$ for some $p > 0$, then r is an integer $\ge p(\alpha + 1)/2\alpha$. Then $b^{1+\alpha}(n)n^{\alpha r} \ge n^{-p(1+\alpha)/2} \to \infty$, and (3.1) is satisfied.

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